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Quantum versus classical decay laws in open chaotic systems

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We study analytically the time evolution in decaying chaotic systems and discuss in detail the hierarchy of characteristic time scales that appeared in the quasiclassical region. There exist two quantum time scales: the Heisenberg time t_H and the time $t_q = t_H/\sqrt{\kappa T}$ (with $\kappa \gg 1$ and T being the degree of resonance overlapping and the transmission coefficient, respectively) associated with the decay. If $t_q < t_H$ the quantum deviation from the classical decay law starts at the time t_q and are due to the openness of the system. Under the opposite condition quantum effects in intrinsic evolution begin to influence the decay at the time t_H. In this case we establish the connection between quantities which describe the time evolution in an open system and their closed counterparts. [S1063-651X(97)50711-2]

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In a recent paper [1] G. Casati *et al.*, using numerical simulations in the kicked rotor model with relaxation, have demonstrated that a new time scale exists for a decaying quantum system in the deep quasiclassical region. After this time, which is much less than the Heisenberg time $t_H = 2\pi\rho$ (with ρ being the mean level density and $\hbar = 1$), the decay law begins to deviate from the classical one.

The aim of the present paper is to show that such a time scale is, in fact, a general feature of open quantum chaotic systems and is related to peculiarities in fluctuations of the resonance widths. We describe these fluctuations in the framework of the random matrix approach and employ the formalism of the effective non-Hermitian Hamiltonian which is commonly used in the theory of resonance scattering [2,3].

Generally, decay properties of open quantum systems are related to fluctuations in complex eigenvalues (resonance energies) $\mathcal{E}_n = E_n - (i/2)\Gamma_n$ of the effective Hamiltonian $\mathcal{H}$ via the two-point correlator of the Green's operator $\mathcal{G}(\mathcal{E}) = (\mathcal{E} - \mathcal{H})^{-1}$. As typical examples one can mention the S-matrix [2,4–6] or time delay [7–10,14] correlation functions. The simplest quantity of such a kind is the leakage of the norm inside an open system

$$P(t) = (1/N)\langle\, \mathrm{Tr}\{\exp(i\mathcal{H}^\dagger t)\exp(-i\mathcal{H}t)\}\rangle, \tag{1}$$

which is a somewhat simplified version of the decay functions considered in Ref. [5]. Here, the angle brackets stand for the random matrix ensemble average, and the equivalence of spectral and ensemble averages is implied [11]. It is easy to see that similar quantity (without the ensemble averaging) has been numerically evaluated in [1]. In the case of a closed system $P(t) \equiv 1$ at any time. The time dependence in Eq. (1) appears due to the anti-Hermitian part of the operator $\mathcal{H}$. The well-known relation between the time evolution operator $\exp(-i\mathcal{H}t)$ and the Green's function $\mathcal{G}(E)$ enables one to represent $P(t)$ in the form of the Fourier integral

$$P(t) = \frac{1}{4\pi^2 N}\int_{-\infty}^{\infty} d\varepsilon\, e^{-i\varepsilon t}\int_{-\infty}^{\infty} dE \left\langle \mathrm{Tr}\mathcal{G}\left(E + \frac{\varepsilon}{2}\right)\mathcal{G}^\dagger\left(E - \frac{\varepsilon}{2}\right)\right\rangle. \tag{2}$$

In the eigenbasis of the effective Hamiltonian the trace in Eq. (1) can be represented in the form

$$P(t) = \frac{1}{N}\left\langle \sum_{n,n'} U_{n'n}^2 \exp\{-i(\mathcal{E}_n - \mathcal{E}_{n'}^*)t\}\right\rangle \tag{3}$$

$$= \int_{-\infty}^{\infty} d\varepsilon\, e^{-i\varepsilon t}\int_0^{\infty} d\Gamma\, e^{-\Gamma t} R(\varepsilon,\Gamma) \tag{4}$$

where $R(\varepsilon,\Gamma)$ denotes

$$R(\varepsilon,\Gamma) = \frac{1}{N}\left\langle \sum_{n,n'} U_{n'n}^2\, \delta[\varepsilon - (E_n - E_{n'})]\,\delta\left[\Gamma - \frac{\Gamma_n + \Gamma_{n'}}{2}\right]\right\rangle$$

and $U_{n'n} = \langle \psi_{n'}|\psi_n\rangle$ is the Bell-Steinberger nonorthogonality matrix [12] of the eigenvectors $|\psi_n\rangle$ of $\mathcal{H}$. This matrix differs from the unity only if resonances overlap. It is worth noting that, contrary to the ε dependence of the function $R(\varepsilon,\Gamma)$ determined by the level spacings along the real energy axis, its Γ dependence is governed by widths themselves. Therefore, the decay law cannot be directly related to the distances $\sqrt{(E_n - E_{n'})^2 + (\Gamma_n - \Gamma_{n'})^2/4}$ between resonance levels in the complex energy plane. This is contrary to what was conjectured in Ref. [1].

Prior to exact calculating $P(t)$ we would like first to perform qualitative analysis. As it will be justified below by the exact calculation, the main features of $P(t)$ can be understood already from calculation of the diagonal part

$$P_d(t) = \frac{1}{N}\left\langle \sum_n e^{-\Gamma_n t}\right\rangle = \int_0^{\infty} d\Gamma\, \mathcal{P}(\Gamma)\, e^{-\Gamma t}. \tag{5}$$

Here we put approximately $U_{nn} = 1$, neglecting its smooth dependence on the index n. The function $\mathcal{P}(\Gamma)$ is the distribution of the resonance widths. This function is explicitly known [13,14] for the case of the unitary ensemble which corresponds to the systems with the broken time-reversal symmetry. Therefore, we use this ensemble to demonstrate our general statements. In Ref. [14] detailed analysis of the width distribution has been performed, in particular, convenient for our purposes integral representation

$$\mathcal{P}(\eta)=\frac{1}{\kappa\eta^2}\frac{1}{(M-1)!}\int_{\eta\kappa(1-T)/T}^{\eta\kappa/T}d\xi e^{-\xi+M\ln\xi} \tag{6}$$

is given, with $\eta=\Gamma/\Gamma_W$ being the decay width measured in the units of the Weisskopf width [15,16]

$$\Gamma_W=MT/2\pi\rho, \tag{7}$$

where T is the transmission coefficient and the dimensionless parameter $\kappa=2\pi\rho\Gamma_W=MT$ characterizes the degree of resonance overlapping. The rate of small widths diminishes rapidly when the number M of (statistically equivalent) open decay channels grows. For small overlapping, $\kappa\ll1$, the density $\mathcal{P}(\eta)$ simplifies to the well-known χ^2_M distribution. However, quasiclassics corresponds to $M\gg1$ and strong overlapping $\kappa\gg1$ [16]. In this case the width distribution decreases exponentially at $\Gamma<\Gamma_W$ and follows the power law $\sim(\Gamma_W/\Gamma)^2$ within some domain above Γ_W. In the classical limit $M,\rho\to\infty$ but Γ_W is kept fixed and identified with the classical escape rate [16], an empty strip appears below the value Γ_W [17,6].

Substituting Eq. (6) in Eq. (5) and changing the order of integration, we come to the expression

$$P_d(t)=\frac{1}{T}\int_0^{T/(1-T)}\frac{d\xi}{(1+\xi)^2}\exp\left\{-M\ln\left[1+\frac{1+\xi}{M}\Gamma_W t\right]\right\}, \tag{8}$$

which is still exact in M and T. In the classical limit defined above the first term in the $1/M$ expansion of the logarithm in Eq. (8) gives $P_d(t)=P_{cl}(t)p(t)$, where $P_{cl}(t)=\exp(-\Gamma_W t)$ is the classical decay probability which follows from the semiclassical periodic orbit theory [19] and $p(t)$ is a slowly varying factor, the proper calculation of which lies beyond the diagonal approximation. Further terms of the $1/M$ expansion can be neglected for the times appreciably less than the characteristic time

$$t_q=\sqrt{M}t_W=\sqrt{\kappa/T}t_W=t_H/\sqrt{\kappa T}, \tag{9}$$

where $t_W\equiv1/\Gamma_W$ is the characteristic lifetime of the system. The quantum time scale t_q is similar to that found in Ref. [1] (see also [20]). In the mesoscopic systems the typical life time is given by the Thouless time [18]. Therefore, the connection $t_H=\kappa t_W$ shows that our overlapping parameter κ plays the role analogous to the dimensionless conductance in the mesoscopic physics. We note in this respect that the ratio t_W/t_q differs from that conjectured in Ref. [1] by an additional factor $\sqrt{T}$, which depends on the strength of coupling to channels. It is also worth noting that the time t_q appears also in the relaxation phenomena in disordered conductors as well [21,22].

The next-to-leading term of the expansion being positive, quantum corrections slow down the decay law at $t>t_q$. After this time crossover occurs to the asymptotic power law

$$P_d^{(as)}(t)=\kappa^{-1}(\Gamma_W t/M)^{-M}, \tag{10}$$

which is characteristic for open quantum systems [16]. As it will be demonstrated below, this expression correctly matches the exact result. The fact that the diagonal approximation properly reproduces the asymptotic behavior was first noted in [23]. One can easily see from Eq. (5) that such a power behavior comes from the influence of the widths which are smaller than Γ_W. Their rate differs from zero as long as the parameter $1/M$ remains finite.

Qualitative arguments presented above can be put on a rigorous ground. Powerful supersymmetry technique [24,2] enables us to perform exact calculations in Eq. (2). Skipping the details of quite a standard calculation, we concentrate on the analysis of the result. For the case of unitary symmetry it reads

$$P(t)=\int_{-1}^{1}d\lambda_0\int_1^{\infty}d\lambda_1\mu(\lambda_i)f(\lambda_i)\delta(t/t_H-(\lambda_1-\lambda_0)/2)\times\left[\frac{1+T(\lambda_0-1)/2}{1+T(\lambda_1-1)/2}\right]^M, \tag{11}$$

where $\mu(\lambda_i)=(\lambda_1-\lambda_0)^{-2}$ is the measure of integration [24] and $f(\lambda_i)=(\lambda_1^2-\lambda_0^2)/2$. The openness of the system is contained in the last "channel" factor in Eq. (11). Actually, the structure of the expression (11) is of universal nature for different quantities that describe the time evolution of a chaotic quantum system. It consists of the integration with the measure which is specific for the chosen ensemble, the channel factor, and the preexponent $f(\lambda_i)$. The latter is the only factor that depends on the concrete quantity considered. Since the time dependence related to decay properties comes just from the channel factor, our analysis is of quite a general meaning.

At $t<t_H$ the decay probability (11) can be represented in the form

$$P(t)=\int_0^1 d\nu[1+(1-2\nu)t/t_H]\times\exp\left[M\ln\left(1-\frac{\Gamma_W t/M}{1+(1-\nu)\Gamma_W t/M}\right)\right], \tag{12}$$

whereas at $t>t_H$ it looks like

$$P(t)=e^{-M\ln(1+\Gamma_W t/M)}\int_0^1 d\nu[1+(1-2\nu)t_H/t]\times\left[\frac{1-\nu T}{1-\nu T/(1+\Gamma_W t/M)}\right]^M. \tag{13}$$

In the classical limit (see above) simple calculation leads to the classical decay law $P_{cl}(t)$.

The peculiarities of quantum deviation from the classical time evolution depend significantly on the ratio $t_H/t_q=\sqrt{\kappa T}$. If $\kappa T\gg1$, so that $t_W\ll t_q\ll t_H$, the analysis of Eq. (12) goes along the similar way as described below Eq. (8) and leads to the same conclusions. In the opposite case $\kappa T\ll1$ (but still $\kappa\gg1$, which implies small values of the transmission coefficient T) one arrives, by inspecting the integral factor in Eq. (13), at the general relation

$$P_{open}(t)=(1+\Gamma_W t/M)^{-M}P_{closed}(t), \tag{14}$$

which remains valid until the time $t_f = t_W/T = t_q/\sqrt{\kappa T} = t_H/\kappa T$. Here $P_{closed}(t)$ is the Fourier transform of the spectral correlation function which describes the time evolution in the corresponding closed system. Parallel supersymmetry calculation of the function $P(t)$ for the case of the orthogonal ensemble (which corresponds to chaotic systems with time-reversal symmetry) gives instead of Eq. (11) an expression of similar structure but with obvious changes which are characteristic to the symmetry class considered. We only mention that the channel factor contains the power $M/2$ rather than M. Therefore, the same relation (14) with M substituted by $M/2$ is valid also for time-reversal invariant systems.

The decay factor $(1+\Gamma_W t/M)^{-M}$ is equivalent to the classical exponent until the time t_q. However, in the taken case the Heisenberg time t_H is smaller than t_q, which yields the influence of quantum effects on the time evolution via the function $P_{closed}(t)$. At last, after the time $\kappa t_f = t_H/T$ the asymptotic regime $P^{(as)}(t) \sim (\Gamma_W t/M)^{-M}$ appears with a proportionality coefficient depending on the quantity considered. In the case of the function (1) its asymptotics coincides with that given by Eq. (10).

In conclusion, even in the quasiclassical domain there exists a finite probability of the widths less than the classical escape rate Γ_W. This leads to the appearance of the new quantum time scale $t_q = \sqrt{\kappa/T}\, t_W = t_H/\sqrt{\kappa T}$ associated with the decay. The parameter of resonance overlapping $\kappa \gg 1$ plays the role analogous to the dimensionless conductance in condensed matter physics. The quantum effects begin to influence the time evolution starting from the time t_q if $t_q/t_H = 1/\sqrt{\kappa T} \ll 1$ and from the Heisenberg time t_H under the opposite condition. In the latter case the relation (14) holds connecting the time evolution in an open system with its closed counterpart.

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